

TWO-PHASE FORWARD SOLUTIONS FOR ONE-DIMENSIONAL FORWARD-BACKWARD PARABOLIC EQUATIONS WITH LINEAR CONVECTION AND REACTION

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ABSTRACT. We study the existence and properties of Lipschitz continuous weak solutions to the Neumann boundary value problem for a class of one-dimensional quasilinear forward-backward diffusion equations with linear convection and reaction. The diffusion flux function is assumed to be of a forward-backward type that contains two forward-diffusion phases. We prove that, for all smooth initial data, there exists at least one weak solution whose spatial derivative stays in the two forward phases. Also, for all smooth initial data that have a derivative value lying in a certain phase transition range, we show that there exist infinitely many such solutions that exhibit instantaneous phase transitions between the two forward phases. Moreover, we introduce the notion of transition gauge for such forward solutions and prove that the gauge of all constructed two-phase forward solutions can be arbitrarily close to a certain fixed constant.

1. INTRODUCTION

In this paper, we study the existence and some properties of solutions for a class of one-dimensional quasilinear diffusion equations of the form

$$(1.1) \quad u_t = (\sigma(u_x))_x + b(x, t)u_x + c(x, t)u + f(x, t) \quad \text{in } \Omega \times (0, T),$$

coupled with the initial and Neumann boundary conditions

$$(1.2) \quad \begin{cases} u = u_0 & \text{on } \Omega \times \{t = 0\}, \\ u_x = 0 & \text{on } \partial\Omega \times (0, T). \end{cases}$$

Here, $\Omega = (0, 1) \subset \mathbb{R}$ is the spatial domain, $T > 0$ is any fixed number, $u_0 = u_0(x) \in C^{2+\alpha}(\bar{\Omega})$, for a given $0 < \alpha < 1$, is an initial datum satisfying the compatibility condition

$$(1.3) \quad u'_0 = 0 \quad \text{on } \partial\Omega,$$

and $u = u(x, t)$ is a solution to problem (1.1)-(1.2) in question.

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The diffusion flux function $\sigma = \sigma(s)$ in (1.1) is assumed to be of a forward-backward type, including the well-known Höllig type [12, 13], that contains two forward-diffusion phases. More precisely, we assume that there exist numbers $s_2 > s_1 > 0$ and $\Lambda \geq \lambda > 0$ such that

$$(1.4) \quad \begin{cases} \sigma \in C^{1+\alpha}((-\infty, s_1] \cup [s_2, \infty)), \\ \sigma' > 0 \text{ in } (-\infty, s_1) \cup (s_2, \infty), \\ \lambda \leq \sigma' \leq \Lambda \text{ in } (-\infty, s_1/2) \cup (2s_2, \infty), \\ \sigma(0) = 0, \text{ and } \sigma(s_1) > \sigma(s_2) \geq 0. \end{cases}$$

Note that σ may not be decreasing or even continuous on $[s_1, s_2]$ and thus $\sigma(s_1)$ and $\sigma(s_2)$ may not be necessarily local extrema. See Figure 1 for a typical graph of such flux functions σ .

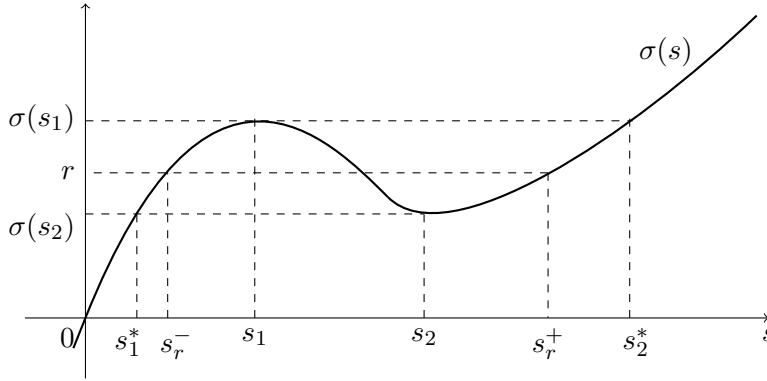


FIGURE 1. A typical graph of flux functions σ . Here, $\sigma(s_1^*) = \sigma(s_2)$, $\sigma(s_2^*) = \sigma(s_1)$ and $\sigma(s_r^\pm) = r$.

Concerning the convection coefficient $b = b(x, t)$, reaction coefficient $c = c(x, t)$ and the source term $f = f(x, t)$ in (1.1), we impose the following conditions:

$$(1.5) \quad \begin{cases} b \in C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T) \cap C^{1,0}(\bar{\Omega}_T), \ c, f \in C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T), \text{ and} \\ c(x, t) = b_x(x, t) + d(t) \ \forall (x, t) \in \bar{\Omega}_T \text{ for some } d \in C([0, T]), \end{cases}$$

where $\Omega_T := \Omega \times (0, T)$. For example, one may choose $b \in C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{\Omega}_T)$, $d \in C^{\frac{\alpha}{2}}([0, T])$ and then take $c = b_x + d(t)$. In particular, when $b = b(t)$, $c = c(t) \in C^{\frac{\alpha}{2}}([0, T])$, condition (1.5) is satisfied.

Equation (1.1) with such a function σ of the type (1.4) arises in the theory of phase transitions in thermodynamics [5, 12, 13, 25]. With different types of nonmonotone flux functions σ , equations of the type (1.1) also stem from mathematical models of stratified turbulent flows [1], population dynamics [21], image processing [22], and gradient flows associated with nonconvex energy functionals [2, 23].

Many theoretical or numerical results on such forward-backward equations or related pseudoparabolic regularizations have been studied in a large

number of literatures; see [1, 2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 26, 27, 28] and references therein. For example, the existence and uniqueness of a so-called *two-phase entropy solution* have been proved in [19] for the Cauchy problem of equation $v_t = (\sigma(v))_{xx}$ with piecewise linear function σ for certain *discontinuous* initial data $v(x, 0)$. Such an equation can be related to equation $u_t = (\sigma(u_x))_x$ by the stream function relation $v = u_x$, $(\sigma(v))_x = u_t$. However, the notion of two-phase entropy solutions and related result cannot handle smooth initial data. On the other hand, for all smooth initial data u_0 , the existence of Lipschitz weak solutions was first established in [28] for problem (1.1)-(1.2) with σ of the type (1.4) and $b = c = f = 0$ in Ω_T , thereby generalizing the result of [12].

In this paper, we prove that for each smooth initial datum u_0 , there exists a Lipschitz weak solution u to problem (1.1)-(1.2) that satisfies $u_x \in (-\infty, s_1) \cup (s_2, \infty)$ a.e. in the space-time domain Ω_T (thus called a *forward solution*) and is smooth in some part of Ω_T near its vertical boundary $\partial\Omega \times [0, T]$. Furthermore, if $s_1^* < u'_0(x_0) < s_2^*$ for some $x_0 \in \Omega$, where $s_1^* \in [0, s_1)$ and $s_2^* \in (s_2, \infty)$ are the unique numbers with $\sigma(s_1^*) = \sigma(s_2)$ and $\sigma(s_2^*) = \sigma(s_1)$ (see Figure 1), then we prove the existence of infinitely many such forward solutions u to (1.1)-(1.2) with the property that

$$(1.6) \quad \begin{cases} |\{(x, t) \in \Omega_T \mid s_1^* < u_x(x, t) < s_1\}| > 0, \\ |\{(x, t) \in \Omega_T \mid s_2 < u_x(x, t) < s_2^*\}| > 0. \end{cases}$$

Therefore, these solutions must exhibit instantaneous phase transitions between the two forward phases, and we will call such solutions *two-phase forward solutions*. In fact, these two-phase forward solutions will also satisfy a much stronger condition in terms of the *transition gauge* defined later – they are *almost transition gauge invariant*.

We remark that property (1.6) is actually strengthened in Theorem 2.2 so that the constructed solutions u satisfy that $(u_x, \sigma(u_x))$ stays in both parts of the set K' defined in (4.1) below on sets of positive measure in Ω_T and thus internally give rise to a *hysteresis loop* [26] that is the boundary of the set U' in (4.1); this is typical in the first-order phase transitions similar to the Ericksen bar problem [6].

The paper is organized as follows. Our main results are stated in Section 2 as Theorem 2.1 and in a more detailed fashion as Theorem 2.2 after the discussion of the main idea of a *nonlocal* differential inclusion and the introduction of *subsolutions* of the inclusion and the notion of *transition gauge* for these subsolutions and for weak solutions of (1.1). In Section 3, a special subsolution w^* is constructed by solving a modified parabolic problem. Section 4 defines a suitable admissible set of subsolutions so that a pivotal density lemma, Lemma 4.1, can be proved later. In Section 5, the proof of Theorem 2.2 is completed in the framework of Baire's category method with the help of Lemma 4.1. As the last part of the paper, the long proof of Lemma 4.1 is given in Section 7 following a technical lemma in Section 6.

In closing this section, we introduce some notations. For a measurable set $E \subset \mathbb{R}^n$ ($n = 1, 2$), we denote by $|E|$ its Lebesgue measure. For a matrix $A = (a_{ij})$ in the space $\mathbb{M}^{2 \times 2}$ of 2×2 real matrices, we use $|A|$ to denote its Hilbert-Schmidt norm, that is, $|A| = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2}$. For a vector $a = (a_1, a_2) \in \mathbb{R}^2$, $|a| = (\sum_{i=1}^2 a_i^2)^{1/2}$ is its Euclidean norm. We mainly follow the notations in the book [17] for function spaces, with one exception that the letter C is used instead of H regarding suitable (parabolic) Hölder spaces. For integers $k, l \geq 0$ with $2l \leq k$, we denote by $C^{k,l}(\bar{\Omega}_T)$ the space of functions $u \in C^0(\bar{\Omega}_T)$ such that $\partial_x^\beta \partial_t^l u \in C^0(\bar{\Omega}_T)$ for all multiindices $|\beta| \leq k$ and integers $0 \leq j \leq l$ with $|\beta| + 2j \leq k$.

2. MAIN RESULTS

To fix some notation, let $s_1^* \in [0, s_1)$ and $s_2^* \in (s_2, \infty)$ denote the unique numbers with $\sigma(s_1^*) = \sigma(s_2)$ and $\sigma(s_2^*) = \sigma(s_1)$, respectively. For each $r \in [\sigma(s_2), \sigma(s_1)]$, let $s_r^+ \in [s_2, s_2^*]$ and $s_r^- \in [s_1^*, s_1]$ be the unique numbers such that $\sigma(s_r^\pm) = r$. (See Figure 1.)

Definition 2.1. A function $u \in W^{1,\infty}(\Omega_T)$ is a *Lipschitz solution* to problem (1.1)-(1.2) provided that equality

$$(2.1) \quad \begin{aligned} \int_0^s \int_0^1 (u\zeta_t - \sigma(u_x)\zeta_x + (bu_x + cu + f)\zeta) dx dt \\ = \int_0^1 (u(x, s)\zeta(x, s) - u_0(x)\zeta(x, 0)) dx \end{aligned}$$

holds for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and each $s \in [0, T]$. A Lipschitz solution u to (1.1)-(1.2) satisfying $u_x \in (-\infty, s_1) \cup (s_2, \infty)$ a.e. in Ω_T will be called a *forward solution*.

Since all solutions u to problem (1.1)-(1.2) considered in this paper will satisfy $u_x \notin (s_2 - \delta_u, s_2 + \delta_u)$ a.e. in Ω_T for some $\delta_u > 0$, we have $u_x = 0$ if and only if $\sigma(u_x) = 0$ even in the case that $\sigma(s_2) = 0$; thus equality (2.1) indeed reflects the Neumann boundary condition in (1.2).

We now state our main existence results in the following simplified version.

Theorem 2.1. *For any initial datum $u_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfying the compatibility condition (1.3), there exists at least one forward solution to problem (1.1)-(1.2). Furthermore, if, in addition, $s_1^* < u'_0(x_0) < s_2^*$ for some $x_0 \in \Omega$, then there are infinitely many forward solutions u satisfying (1.6).*

The first part of this theorem follows easily from its second part. For example, if the initial datum u_0 satisfies $u'_0(x_1) > s_1^*$ for some $x_1 \in \Omega$, then by (1.3) there must be a point $x_0 \in \Omega$ such that $u'_0(x_0) \in (s_1^*, s_2^*)$; thus, in this case, the second part of the theorem implies the existence of forward solutions to problem (1.1)-(1.2). Next, assume that $u'_0(x) \leq s_1^*$ for all $x \in \Omega$. We then solve problem (1.1)-(1.2), with σ replaced by a function $\bar{\sigma}$ on \mathbb{R} whose derivative always lies in between two positive numbers such that

it agrees with σ on $(-\infty, (s_1^* + s_1)/2]$, to get a classical solution u in Ω_T . If $u_x(x, t) \leq s_1^*$ for all $(x, t) \in \Omega_T$, then u itself is a (classical) forward solution to problem (1.1)-(1.2) with the original σ . Otherwise, due to the Neumann boundary condition, we can choose a $0 < T' < T$ such that $u_x(x, t) \leq (s_1^* + s_1)/2$ for all $(x, t) \in \Omega \times (0, T')$ and that $u_x(x_0, t_0) \in (s_1^*, (s_1^* + s_1)/2)$ for some $(x_0, t_0) \in \Omega \times (0, T')$. Then we use $u(\cdot, t_0)$ as an initial datum and apply the second part of the theorem to get forward solutions \tilde{u} to (1.1)-(1.2) in $\Omega \times (t_0, T)$ with the original σ . Piecing u and \tilde{u} together at $t = t_0$, we obtain forward solutions to (1.1)-(1.2) in Ω_T with the original σ .

The second part of Theorem 2.1 follows from our more detailed statement, Theorem 2.2, to be given below. For this, we first recast equation (1.1) as a nonlocal differential inclusion and introduce the concept of subsolutions and transition gauge.

2.1. Nonlocal differential inclusion. Our key approach to problem (1.1)-(1.2) is to treat equation (1.1) as a new type of partial differential inclusions involving a *nonlocal* operator.

To ascertain this treatment, we first let

$$F(x, t) := \int_0^x f(y, t) dy$$

and rewrite equation (1.1) as

$$u_t = (\sigma(u_x) + bu + \mathcal{P}_u + F)_x,$$

where the *potential operator* \mathcal{P} at u is defined by

$$\mathcal{P}_u(x, t) := \int_0^x (c(y, t) - b_y(y, t))u(y, t) dy \quad \forall (x, t) \in \Omega_T.$$

Therefore, $u \in W^{1,\infty}(\Omega_T)$ is a solution of (1.1) in the sense of distributions if and only if there exists a stream function $v \in W^{1,\infty}(\Omega_T)$ satisfying

$$(2.2) \quad v_x = u, \quad v_t = \sigma(u_x) + bu + \mathcal{P}_u + F \quad \text{a.e. in } \Omega_T;$$

such a function v is necessarily unique up to a constant.

Next, let $\Sigma_0 = \{(s, \sigma(s)) \in \mathbb{R}^2 \mid s \in \mathbb{R}\}$, which is simply the graph of the function σ . For each $(x, t) \in \Omega_T$ and each $u \in W^{1,\infty}(\Omega_T)$, define the matrix set

$$\Sigma((x, t); u) = \left\{ \begin{pmatrix} s & c \\ u(x, t) & q \end{pmatrix} \in \mathbb{M}^{2 \times 2} \mid \begin{array}{l} c \in \mathbb{R}, (s, q - b(x, t)u(x, t)) \\ -\mathcal{P}_u(x, t) - F(x, t) \in \Sigma_0 \end{array} \right\}.$$

Then for a vector function $w = (u, v) \in W^{1,\infty}(\Omega_T; \mathbb{R}^2)$, system (2.2) is equivalent to the following differential inclusion:

$$(2.3) \quad \nabla w(x, t) = \begin{pmatrix} u_x(x, t) & u_t(x, t) \\ v_x(x, t) & v_t(x, t) \end{pmatrix} \in \Sigma((x, t); u), \quad \text{a.e. } (x, t) \in \Omega_T.$$

In this manner, (1.1) becomes equivalent to differential inclusion (2.3).

Remark 2.1. We remark that the matrix sets in differential inclusion (2.3) are depending not only on the space-time variable (x, t) but also on the input function u in a nonlocal fashion. So we may call such an inclusion a *nonlocal* partial differential inclusion, which, to our best knowledge, has not been studied in the well-known literatures on partial differential inclusion problems [4, 20]; however, see [24] for a treatment of a nonlocal problem using differential inclusion.

2.2. Subolutions and transition gauge. Let

$$K_0 = \{(s, \sigma(s)) \in \mathbb{R}^2 \mid s \in [s_1^*, s_1] \cup [s_2, s_2^*]\},$$

$$U_0 = \{(s, r) \in \mathbb{R}^2 \mid \sigma(s_2) < r < \sigma(s_1), s_r^- < s < s_r^+\}.$$

A function $w = (u, v) \in W^{1,\infty}(\Omega_T; \mathbb{R}^2)$ is called a *subsolution* of differential inclusion (2.3) if

$$v_x = u \text{ and } (u_x, v_t - bu - \mathcal{P}_u - F) \in \Sigma_0 \cup U_0 \text{ a.e. in } \Omega_T.$$

In this case, we define the *transition set* of w by

$$(2.4) \quad O^w = \{(x, t) \in \Omega_T \mid (u_x, v_t - bu - \mathcal{P}_u - F) \in K_0 \cup U_0\}.$$

Note that the transition set O^w may be a null set; that is, $|O^w| = 0$.

Let $w = (u, v)$ be a subsolution of (2.3). Define a function $Z_w: O^w \rightarrow [0, 1]$ by

$$(2.5) \quad Z_w(x, t) = \frac{u_x(x, t) - S_w^-(x, t)}{S_w^+(x, t) - S_w^-(x, t)},$$

where

$$S_w^\pm(x, t) = s_{v_t(x, t) - b(x, t)u(x, t) - \mathcal{P}_u(x, t) - F(x, t)}^\pm;$$

then, for each non-null subset E of O^w , we define the *transition gauge* of w on E by

$$(2.6) \quad \Gamma_w^E = \frac{1}{|E|} \int_E Z_w(x, t) dx dt.$$

If E is any measurable subset of Ω_T with $|E| = 0$ or $E \subset \Omega_T \setminus O^w$, we define $\Gamma_w^E = -1$. In case of a non-null set $E \subset O^w$, note that $\Gamma_w^E \in [0, 1]$ and that $\Gamma_w^E = 1$ (0, resp.) if and only if $(u_x, v_t - bu - \mathcal{P}_u - F)$ belongs to the right-branch (left-branch, resp.) of K_0 a.e. in E . Therefore, Γ_w^E measures the tendency of phases of the subsolution w towards the right-branch of K_0 on E .

A function $u \in W^{1,\infty}(\Omega_T)$ is called a *subsolution* of equation (1.1) if there exists a function $v \in W^{1,\infty}(\Omega_T)$ so that $w = (u, v)$ is a subsolution of (2.3).

Note that if $u \in W^{1,\infty}(\Omega_T)$ is a solution of (1.1) in the sense of distributions, then, with the stream function $v \in W^{1,\infty}(\Omega_T)$ defined by (2.2) (unique up to a constant), the function $w = (u, v)$ is a solution and thus a subsolution of (2.3). In this case, we define the *transition gauge* of u on any measurable set $E \subset \Omega_T$ by

$$(2.7) \quad \gamma_u^E = \Gamma_w^E.$$

Remark 2.2. For the purpose of this paper, the transition gauge γ_u^E is defined only for (weak) solutions u of equation (1.1), not for general subsolutions of (1.1). In fact, in what follows, we only deal with the transition gauge Γ_w^E for subsolutions w of differential inclusion (2.3) with $|O^w| > 0$ on non-null sets $E \subset O^w$.

2.3. Two-phase forward solutions with almost transition gauge invariance. Assume that the initial datum $u_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfies the compatibility condition (1.3) and the transition condition

$$(2.8) \quad s_1^* < u'_0(x_0) < s_2^* \quad \text{for some } x_0 \in \Omega.$$

Then our detailed version for the main result, Theorem 2.1, can be formulated as follows.

Theorem 2.2. *Let r_1, r_2 be any two numbers with $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$ and $s_{r_1}^- < u'_0(x_0) < s_{r_2}^+$. Then there exists a subsolution $w^* = (u^*, v^*) \in C^{2,1}(\bar{\Omega}_T) \times C^{3,1}(\bar{\Omega}_T)$ of differential inclusion (2.3) such that the sets*

$$\begin{aligned} \Omega_T^1 &= \{(x, t) \in \Omega_T \mid u_x^*(x, t) < s_{r_1}^-\}, \quad \Omega_T^3 = \{(x, t) \in \Omega_T \mid u_x^*(x, t) > s_{r_2}^+\}, \\ \Omega_T^2 &= \{(x, t) \in \Omega_T \mid s_{r_1}^- < u_x^*(x, t) < s_{r_2}^+\}, \\ \Omega_T^{0,-} &= \{(x, t) \in \Omega_T \mid u_x^*(x, t) = s_{r_1}^-\}, \quad \Omega_T^{0,+} = \{(x, t) \in \Omega_T \mid u_x^*(x, t) = s_{r_2}^+\} \end{aligned}$$

satisfy $E := \Omega_T^2 \neq \emptyset$, $0 < \Gamma_{w^*}^E < 1$, and

$$(2.9) \quad \begin{cases} ((0, \delta^*) \cup (1 - \delta^*, 1)) \times (0, T) \subset \Omega_T^1 \text{ for some } \delta^* > 0, \\ \partial \Omega_T^1 \cap \{(x, 0) \mid x \in \Omega\} = \{(x, 0) \mid x \in \Omega, u'_0(x) < s_{r_1}^-\}, \\ \partial \Omega_T^2 \cap \{(x, 0) \mid x \in \Omega\} = \{(x, 0) \mid x \in \Omega, s_{r_1}^- < u'_0(x) < s_{r_2}^+\}, \\ \partial \Omega_T^3 \cap \{(x, 0) \mid x \in \Omega\} = \{(x, 0) \mid x \in \Omega, u'_0(x) > s_{r_2}^+\}. \end{cases}$$

Furthermore, for each $\epsilon > 0$, there exist infinitely many Lipschitz solutions u to problem (1.1)-(1.2) satisfying the following properties:

- (a) $u = u^*$ in $\Omega_T^1 \cup \Omega_T^3$, $\|u - u^*\|_{L^\infty(\Omega_T)} < \epsilon$, $\|u_t - u_t^*\|_{L^\infty(\Omega_T)} < \epsilon$,
- (b) $u_x < s_{r_1}^-$ in Ω_T^1 , $u_x \in [s_{r_1}^-, s_{r_2}^-] \cup [s_{r_1}^+, s_{r_2}^+]$ a.e. in Ω_T^2 , $u_x > s_{r_2}^+$ in Ω_T^3 ,
- (c) $u_x = s_{r_1}^-$ a.e. in $\Omega_T^{0,-}$, $u_x = s_{r_2}^+$ a.e. in $\Omega_T^{0,+}$,
- (d) $|F_T^+| = \gamma_u^E |\Omega_T^2|$, $|F_T^-| = (1 - \gamma_u^E) |\Omega_T^2|$, and $|\gamma_u^E - \Gamma_{w^*}^E| < \epsilon$, where $F_T^\pm := \{(x, t) \in \Omega_T^2 \mid u_x(x, t) \in [s_{r_1}^\pm, s_{r_2}^\pm]\}$.

Clearly, this theorem implies the second part of Theorem 2.1 if the number ϵ is chosen so that

$$0 < \epsilon < \min\{\Gamma_{w^*}^E, 1 - \Gamma_{w^*}^E\};$$

thus such functions u are two-phase forward solutions to problem (1.1)-(1.2).

The proof of the first part of Theorem 2.2 is given in the next section; that of the second part will be given in Section 5.

3. CONSTRUCTION OF SUBSOLUTION BY A MODIFIED PROBLEM

In this section, we assume that $u_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfies conditions (1.3) and (2.8). We also assume that r_1, r_2 are any two numbers such that

$$\sigma(s_2) < r_1 < r_2 < \sigma(s_1), \quad s_{r_1}^- < u'_0(x_0) < s_{r_2}^+.$$

3.1. Modified problem. By elementary calculus with (1.4), we can construct a function $\tilde{\sigma} \in C^{1+\alpha}(\mathbb{R})$ satisfying

$$(3.1) \quad \begin{cases} \tilde{\sigma} = \sigma & \text{on } (-\infty, s_{r_1}^-] \cup [s_{r_2}^+, \infty), \\ \tilde{\sigma} < \sigma & \text{on } (s_{r_1}^-, s_{r_2}^-], \tilde{\sigma} > \sigma & \text{on } [s_{r_1}^+, s_{r_2}^+), \text{ and} \\ \tilde{\lambda} \leq \tilde{\sigma}' \leq \tilde{\Lambda} & \text{in } \mathbb{R}, \end{cases}$$

where $\tilde{\Lambda} \geq \tilde{\lambda} > 0$ are some constants. (See Figure 2.)

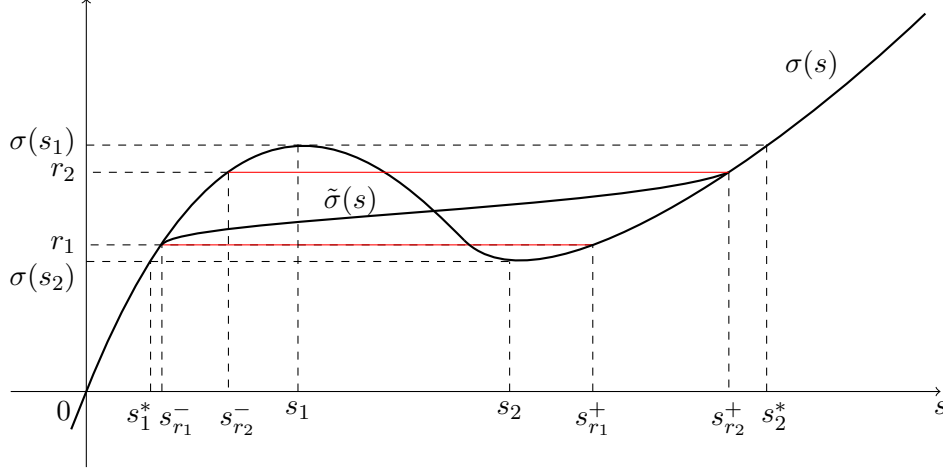


FIGURE 2. The modified function $\tilde{\sigma}$, constructed as in (3.1), is strictly increasing on \mathbb{R} and agrees with σ on $(-\infty, s_{r_1}^-] \cup [s_{r_2}^+, \infty)$. The set K' defined in (4.1) below consists of the two pieces of the graph of σ over $[s_{r_1}^-, s_{r_2}^-]$ and $[s_{r_1}^+, s_{r_2}^+]$, while the set U' in (4.1) is the open set bounded by K' and the two parallel line segments (in red) joining the endpoints of K' . The boundary $\partial U'$ of U' represents a hysteresis loop.

From the standard parabolic theory [18, Theorem 13.24], we obtain a unique classical solution $u^* \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}_T)$ to the initial-boundary value problem

$$(3.2) \quad \begin{cases} u_t^* = (\tilde{\sigma}(u_x^*))_x + b(x, t)u_x^* + c(x, t)u^* + f(x, t) & \text{in } \Omega_T, \\ u^* = u_0 & \text{on } \Omega \times \{t = 0\}, \\ u_x^* = 0 & \text{on } \partial\Omega \times (0, T). \end{cases}$$

We then have from (1.5) that

$$(3.3) \quad \begin{aligned} (\tilde{\sigma}(u_x^*) + bu^* + \mathcal{P}_{u^*} + F)_x &= \tilde{\sigma}(u_x^*)_x + b_x u^* + bu_x^* + d(t)u^* + f \\ &= \tilde{\sigma}(u_x^*)_x + bu_x^* + cu^* + f = u_t^* \quad \text{in } \Omega_T. \end{aligned}$$

3.2. Separation of domain. Define the subsets of Ω_T by

$$(3.4) \quad \begin{aligned} \Omega_T^1 &= \{(x, t) \in \Omega_T \mid u_x^*(x, t) < s_{r_1}^-\}, \\ \Omega_T^2 &= \{(x, t) \in \Omega_T \mid s_{r_1}^- < u_x^*(x, t) < s_{r_2}^+\}, \\ \Omega_T^3 &= \{(x, t) \in \Omega_T \mid u_x^*(x, t) > s_{r_2}^+\}, \\ \Omega_T^{0,-} &= \{(x, t) \in \Omega_T \mid u_x^*(x, t) = s_{r_1}^-\}, \\ \Omega_T^{0,+} &= \{(x, t) \in \Omega_T \mid u_x^*(x, t) = s_{r_2}^+\}. \end{aligned}$$

Note that $\Omega_T^2 \neq \emptyset$ as $s_{r_1}^- < u'_0(x_0) < s_{r_2}^+$. From (3.2), we easily have

$$(3.5) \quad \begin{cases} ((0, \delta^*) \cup (1 - \delta^*, 1)) \times (0, T) \subset \Omega_T^1 \text{ for some } \delta^* > 0, \\ \partial\Omega_T^1 \cap \{(x, 0) \mid x \in \Omega\} = \{(x, 0) \mid x \in \Omega, u'_0(x) < s_{r_1}^-\}, \\ \partial\Omega_T^2 \cap \{(x, 0) \mid x \in \Omega\} = \{(x, 0) \mid x \in \Omega, s_{r_1}^- < u'_0(x) < s_{r_2}^+\}, \\ \partial\Omega_T^3 \cap \{(x, 0) \mid x \in \Omega\} = \{(x, 0) \mid x \in \Omega, u'_0(x) > s_{r_2}^+\}. \end{cases}$$

3.3. Construction of subsolution. Define an *auxiliary function* v^* by

$$\begin{aligned} v^*(x, t) &= \int_0^t (\tilde{\sigma}(u_x^*(x, \tau)) + b(x, \tau)u^*(x, \tau) + \mathcal{P}_{u^*}(x, \tau) + F(x, \tau)) d\tau \\ &\quad + \int_0^x u^*(y, 0) dy \quad \forall (x, t) \in \Omega_T. \end{aligned}$$

From (3.3), the function $w^* := (u^*, v^*)$ satisfies that in Ω_T ,

$$(3.6) \quad \begin{cases} v_t^* - bu^* - \mathcal{P}_{u^*} - F = \tilde{\sigma}(u_x^*), \\ v_x^* = u^*, \end{cases}$$

and hence from (3.1), $w^* \in C^{2,1}(\bar{\Omega}_T) \times C^{3,1}(\bar{\Omega}_T)$ is a subsolution of (2.3). Clearly, $E = \Omega_T^2 \subset O^{w^*}$, and it follows from (3.6) that the number $\Gamma_{w^*}^E$ is given by

$$\Gamma_{w^*}^E = \frac{1}{|\Omega_T^2|} \int_{\Omega_T^2} \frac{u_x^* - s_{\tilde{\sigma}(u_x^*)}^-}{s_{\tilde{\sigma}(u_x^*)}^+ - s_{\tilde{\sigma}(u_x^*)}^-} dx dt,$$

which lies in $(0, 1)$.

4. ADMISSIBLE SET AND DENSITY LEMMA

We use the same notation as in the previous section and, for simplicity, in what follows, we say that an open set $G \subset \mathbb{R}^2$ is *regular* if $|\partial G| = 0$.

4.1. Related matrix sets. Define the sets (see Figure 2)

$$(4.1) \quad \begin{aligned} K' &= \{(s, \sigma(s)) \in \mathbb{R}^2 \mid s \in [s_{r_1}^-, s_{r_2}^-] \cup [s_{r_1}^+, s_{r_2}^+]\}, \\ U' &= \{(s, r) \in \mathbb{R}^2 \mid r_1 < r < r_2, s_r^- < s < s_r^+\}. \end{aligned}$$

For each $(x, t) \in \Omega_T$ and each $u \in W^{1,\infty}(\Omega_T)$, define the matrix sets

$$\begin{aligned} K((x, t); u) &= \left\{ \begin{pmatrix} s & c \\ u(x, t) & q \end{pmatrix} \in \mathbb{M}^{2 \times 2} \mid \begin{array}{l} |c| \leq m^*, (s, q - b(x, t)u(x, t)) \\ -\mathcal{P}_u(x, t) - F(x, t) \in K' \end{array} \right\}, \\ U((x, t); u) &= \left\{ \begin{pmatrix} s & c \\ u(x, t) & q \end{pmatrix} \in \mathbb{M}^{2 \times 2} \mid \begin{array}{l} |c| < m^*, (s, q - b(x, t)u(x, t)) \\ -\mathcal{P}_u(x, t) - F(x, t) \in U' \end{array} \right\}, \end{aligned}$$

where $m^* := \|u_t^*\|_{L^\infty(\Omega_T)} + 1 > 0$.

Note that $K((x, t); u)$ is a compact subset of $\mathbb{M}^{2 \times 2}$ for each $(x, t) \in \Omega_T$ and each $u \in W^{1,\infty}(\Omega_T)$.

4.2. Admissible set of subsolutions. For any fixed $\epsilon > 0$, we define a set \mathcal{A} of *admissible functions* by

$$\mathcal{A} = \left\{ \begin{array}{l} w = (u, v) \text{ belongs to } C^{2,1}(\bar{\Omega}_T) \times C^{3,1}(\bar{\Omega}_T) \\ \left| \begin{array}{l} w = w^* \text{ in } \Omega_T \setminus \bar{\Omega}_T^w \text{ for some regular} \\ \text{open set } \Omega_T^w \subset \subset \Omega_T^2 = E, \\ \|u - u^*\|_\infty < \epsilon/2, \|u_t - u_t^*\|_\infty < \epsilon/2, \\ \nabla w(x, t) \in U((x, t); u) \ \forall (x, t) \in \Omega_T^2, \\ \mathcal{P}_u = \mathcal{P}_{u^*} \text{ in } \Omega_T \setminus \bar{\Omega}_T^w, |\Gamma_w^E - \Gamma_{w^*}^E| < \epsilon/2 \end{array} \right. \end{array} \right\},$$

where we use $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega_T)}$ here and below. We easily see that $\mathcal{A} \neq \emptyset$ as $w^* \in \mathcal{A}$ and that each function w in \mathcal{A} is a subsolution of (2.3) with its transition set O^w containing $E = \Omega_T^2$.

For each $\delta > 0$, we define the set \mathcal{A}_δ of δ -*approximating functions* by

$$\mathcal{A}_\delta = \left\{ w \in \mathcal{A} \mid \int_{\Omega_T^2} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dxdt \leq \delta |\Omega_T^2| \right\}.$$

4.3. Density lemma. The completion of the proof of Theorem 2.2 relies on the following pivotal density lemma.

Lemma 4.1 (Density Lemma). *For each $\delta > 0$, \mathcal{A}_δ is dense in \mathcal{A} under the L^∞ -norm.*

The proof of this lemma is long and will be given in the last part of the paper, Section 7.

5. PROOF OF THEOREM 2.2

We continue to use the same notation as above. The first part of Theorem 2.2 has already been proved in Section 3. In this section we carry out the remaining part of the proof in the functional framework of Baire's category method based on Lemma 4.1.

5.1. Baire's category method. Let \mathcal{X} denote the closure of \mathcal{A} in the space $L^\infty(\Omega_T; \mathbb{R}^2)$; then (\mathcal{X}, L^∞) is a nonempty complete metric space. It is clear that \mathcal{A} is a bounded subset of $w^* + W_0^{1,\infty}(\Omega_T; \mathbb{R}^2)$, and so $\mathcal{X} \subset w^* + W_0^{1,\infty}(\Omega_T; \mathbb{R}^2)$.

Since the space-time gradient operator $\nabla : \mathcal{X} \rightarrow L^1(\Omega_T; \mathbb{M}^{2 \times 2})$ is a Baire-one map [3, Proposition 10.17], it follows from the Baire Category Theorem [3, Theorem 10.15] that the set of points of discontinuity of the operator ∇ , say \mathcal{D}_∇ , is a set of the first category; thus the set of points at which ∇ is continuous, that is, $\mathcal{C}_\nabla := \mathcal{X} \setminus \mathcal{D}_\nabla$, is dense in \mathcal{X} .

We now proceed to complete the proof of Theorem 2.2 assuming the validity of Lemma 4.1.

Proposition 5.1. *Let $w = (u, v) \in \mathcal{C}_\nabla$. Then u is a Lipschitz solution to problem (1.1)–(1.2) satisfying (a)–(d) of Theorem 2.2.*

Proof. By the density of \mathcal{A} in \mathcal{X} , we can choose a sequence $\{\tilde{w}_j\}_{j \in \mathbb{N}}$ in \mathcal{A} so that $\|\tilde{w}_j - w\|_\infty \rightarrow 0$ as $j \rightarrow \infty$. By Lemma 4.1, for each $j \in \mathbb{N}$, we can choose a function $w_j = (u_j, v_j) \in \mathcal{A}_{1/j}$ with $\|w_j - \tilde{w}_j\|_\infty < 1/j$. As $w_j \rightarrow w$ in \mathcal{X} and $w \in \mathcal{C}_\nabla$, we have that $\nabla w_j \rightarrow \nabla w$ in $L^1(\Omega_T; \mathbb{M}^{2 \times 2})$ and thus a.e. in Ω_T after passing to a subsequence if necessary.

Let $j \in \mathbb{N}$. Since $w_j \in \mathcal{A}_{1/j} \subset \mathcal{A}$, we have the following:

$$(5.1) \quad \begin{cases} w_j = w^* \text{ in } \Omega_T \setminus \bar{\Omega}_T^{w_j} \text{ for some regular open set } \Omega_T^{w_j} \subset \subset \Omega_T^2, \\ \nabla w_j(x, t) \in U((x, t); u_j) \quad \forall (x, t) \in E = \Omega_T^2, \\ \mathcal{P}_{u_j} = \mathcal{P}_{u^*} \text{ in } \Omega_T \setminus \bar{\Omega}_T^{w_j}, \\ \|u_j - u^*\|_\infty < \frac{\epsilon}{2}, \|(u_j)_t - u_t^*\|_\infty < \frac{\epsilon}{2}, |\Gamma_{w_j}^E - \Gamma_{w^*}^E| < \frac{\epsilon}{2}, \text{ and} \\ \int_{\Omega_T^2} \text{dist}(\nabla w_j(x, t), K((x, t); u_j)) \, dxdt \leq \frac{1}{j} |\Omega_T^2|. \end{cases}$$

From this and (3.6), we easily see that

$$(5.2) \quad \begin{cases} (u_j)_x = s_{r_1}^- \text{ in } \Omega_T^{0,-}, (u_j)_x = s_{r_2}^+ \text{ in } \Omega_T^{0,+}, \\ u_j = u^* \text{ in } \Omega_T^1 \cup \Omega_T^3, \text{ and} \\ (v_j)_x = u_j \text{ in } \Omega_T. \end{cases}$$

Also, from (3.1), (3.6), (5.1) and the definition of Ω_T^2 , it follows that in $\Omega_T \setminus \Omega_T^2$,

$$(v_j)_t - bu_j - \mathcal{P}_{u_j} - F = v_t^* - bu^* - \mathcal{P}_{u^*} - F = \tilde{\sigma}(u_x^*) = \sigma((u_j)_x).$$

Now, let $j \rightarrow \infty$; since $\nabla w_j \rightarrow \nabla w$ a.e. in Ω_T , we have

$$(5.3) \quad \begin{cases} u_x = s_{r_1}^- \text{ a.e. in } \Omega_T^{0,-}, u_x = s_{r_2}^+ \text{ a.e. in } \Omega_T^{0,+}, \\ u = u^* \text{ in } \Omega_T^1 \cup \Omega_T^3, \quad v_x = u \text{ a.e. in } \Omega_T, \\ \|u - u^*\|_\infty \leq \frac{\epsilon}{2}, \|u_t - u_t^*\|_\infty \leq \frac{\epsilon}{2}, |\Gamma_w^E - \Gamma_{w^*}^E| \leq \frac{\epsilon}{2}, \text{ and} \\ v_t - bu - \mathcal{P}_u - F = \sigma(u_x) \text{ a.e. in } \Omega_T \setminus \Omega_T^2. \end{cases}$$

Here, we see that (a) and (c) of Theorem 2.2 are satisfied. Also, it follows from (5.1), the continuity of the distance function and the Dominated

Convergence Theorem that

$$\nabla w(x, t) \in K((x, t); u) \quad \text{a.e. } (x, t) \in \Omega_T^2.$$

This inclusion implies that a.e. in Ω_T^2 ,

$$(5.4) \quad \begin{cases} u_x \in [s_{r_1}^-, s_{r_2}^-] \cup [s_{r_1}^+, s_{r_2}^+], \\ v_t - bu - \mathcal{P}_u - F = \sigma(u_x). \end{cases}$$

Thus (b) of Theorem 2.2 follows from (5.3), (5.4) and the definition of Ω_T^1 and Ω_T^3 . From (5.1), (5.3) and (5.4), we have

$$(5.5) \quad \mathcal{P}_u = \mathcal{P}_{u^*} \quad \text{on } \bar{\Omega}_T \setminus \Omega_T^2, \quad v_t - bu - \mathcal{P}_u - F = \sigma(u_x) \quad \text{a.e. in } \Omega_T.$$

From (5.4) and the validity of (b), we also have

$$\Gamma_w^E = \frac{1}{|\Omega_T^2|} \int_{F_T^+ \cup F_T^-} \frac{u_x - s_{\sigma(u_x)}^-}{s_{\sigma(u_x)}^+ - s_{\sigma(u_x)}^-} dx dt = \frac{|F_T^+|}{|\Omega_T^2|};$$

that is,

$$|F_T^+| = \Gamma_w^E |\Omega_T^2|, \quad |F_T^-| = (1 - \Gamma_w^E) |\Omega_T^2|,$$

where $|\Gamma_w^E - \Gamma_{w^*}^E| \leq \frac{\epsilon}{2} < \epsilon$. Since u is a Lipschitz solution of equation (1.1) as we check below, it follows from (5.3), (5.5) and the definition of γ_u^E in (2.7) that $\gamma_u^E = \Gamma_w^E$. Thus (d) holds.

Finally, to verify that u is a Lipschitz solution to (1.1)-(1.2), let $j \in \mathbb{N}$, $\zeta \in C^\infty(\bar{\Omega}_T)$ and $s \in [0, T]$. Note from (5.2) and $w_j = w^*$ on $\partial\Omega_T$ that

$$\begin{aligned} \int_0^s \int_0^1 u_j \zeta_t dx dt &= \int_0^s \int_0^1 (v_j)_x \zeta_t dx dt \\ &= - \int_0^s \int_0^1 (v_j)_{xt} \zeta dx dt + \int_0^1 ((v_j)_x(x, s) \zeta(x, s) - (v_j)_x(x, 0) \zeta(x, 0)) dx \\ &= \int_0^s \int_0^1 (v_j)_t \zeta_x dx dt - \int_0^s ((v_j)_t(1, t) \zeta(1, t) - (v_j)_t(0, t) \zeta(0, t)) dt \\ &\quad + \int_0^1 ((v_j)_x(x, s) \zeta(x, s) - (v_j)_x(x, 0) \zeta(x, 0)) dx \\ &= \int_0^s \int_0^1 (v_j)_t \zeta_x dx dt - \int_0^s (v_t^*(1, t) \zeta(1, t) - v_t^*(0, t) \zeta(0, t)) dt \\ &\quad + \int_0^1 (u_j(x, s) \zeta(x, s) - u_0(x) \zeta(x, 0)) dx. \end{aligned}$$

Let $j \rightarrow \infty$; then by (1.5) and (5.5),

$$\begin{aligned}
\int_0^s \int_0^1 u \zeta_t dx dt &= \int_0^s \int_0^1 v_t \zeta_x dx dt - \int_0^s (v_t^*(1, t) \zeta(1, t) - v_t^*(0, t) \zeta(0, t)) dt \\
&\quad + \int_0^1 (u(x, s) \zeta(x, s) - u_0(x) \zeta(x, 0)) dx \\
&= \int_0^s \int_0^1 (bu + \mathcal{P}_u + F + \sigma(u_x)) \zeta_x dx dt \\
&\quad - \int_0^s (v_t^*(1, t) \zeta(1, t) - v_t^*(0, t) \zeta(0, t)) dt + \int_0^1 (u(x, s) \zeta(x, s) - u_0(x) \zeta(x, 0)) dx \\
&= \int_0^s \int_0^1 (\sigma(u_x) \zeta_x - (bu_x + cu + f) \zeta) dx dt + \int_0^1 (u(x, s) \zeta(x, s) - u_0(x) \zeta(x, 0)) dx,
\end{aligned}$$

where the last equality comes from the observation through (3.6) that

$$\begin{aligned}
&\int_0^s (v_t^*(1, t) \zeta(1, t) - v_t^*(0, t) \zeta(0, t)) dt \\
&= \int_0^s \left((b(1, t) u^*(1, t) + \mathcal{P}_{u^*}(1, t) + F(1, t) + \tilde{\sigma}(u_x^*(1, t))) \zeta(1, t) \right. \\
&\quad \left. - (b(0, t) u^*(0, t) + \mathcal{P}_{u^*}(0, t) + F(0, t) + \tilde{\sigma}(u_x^*(0, t))) \zeta(0, t) \right) dt \\
&= \int_0^s \left((b(1, t) u(1, t) + \mathcal{P}_u(1, t) + F(1, t)) \zeta(1, t) \right. \\
&\quad \left. - (b(0, t) u(0, t) + \mathcal{P}_u(0, t) + F(0, t)) \zeta(0, t) \right) dt.
\end{aligned}$$

Thus u satisfies (2.1) and hence is a Lipschitz solution to (1.1)-(1.2). \square

5.2. Completion of proof of Theorem 2.2. Having shown that the first component u of each function $w = (u, v) \in \mathcal{C}_\nabla$ is a Lipschitz solution to problem (1.1)-(1.2) satisfying (a)–(d), it remains to verify that \mathcal{C}_∇ contains infinitely many functions and that no two different functions in \mathcal{C}_∇ have the first component that are equal. First, suppose on the contrary that \mathcal{C}_∇ has only finitely many functions. Then

$$\mathcal{C}_\nabla = \bar{\mathcal{C}}_\nabla = \mathcal{X} = \bar{\mathcal{A}} = \mathcal{A} \ni w^* = (u^*, v^*).$$

By the above argument, u^* becomes a Lipschitz solution to (1.1)-(1.2) satisfying (a)–(d); this is clearly a contradiction. Therefore, \mathcal{C}_∇ contains infinitely many functions. Next, let $w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in \mathcal{C}_\nabla$. It suffices to show that

$$u_1 = u_2 \text{ in } \Omega_T \iff v_1 = v_2 \text{ in } \Omega_T.$$

If $u_1 = u_2$ in Ω_T , then by (5.3), we have $(v_1)_x = u_1 = u_2 = (v_2)_x$ a.e. in Ω_T . As $v_1 = v_2 = v^*$ on $\partial\Omega_T$, we thus have $v_1 = v_2$ in Ω_T . The converse is also easy to show; we omit this.

The proof of Theorem 2.2 is now complete, of course, upon assuming the validity of Lemma 4.1.

6. A TECHNICAL LEMMA

In this section, we equip with an important but technical tool for local patching to be used in the proof of the density lemma, Lemma 4.1. The following result is a refinement of the $(1+1)$ -dimensional version of a combination of [16, Theorem 2.3 and Lemma 4.5].

Lemma 6.1. *Let $Q = (x_1, x_2) \times (t_1, t_2) \subset \mathbb{R}^2$ be an open rectangle, where $x_2 > x_1$ and $t_2 > t_1$ are fixed reals. Given any $\lambda_1 > 0$, $\lambda_2 > 0$ and $\epsilon > 0$, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(Q; \mathbb{R}^2)$ such that*

- (a) $\|\omega\|_{L^\infty(Q)} < \epsilon$, $\|\varphi_t\|_{L^\infty(Q)} < \epsilon$, $\|\psi_t\|_{L^\infty(Q)} < \epsilon$,
- (b) $-\lambda_1 \leq \varphi_x \leq \lambda_2$ in Q ,
- (c) $\begin{cases} \left| |\{(x, t) \in Q \mid \varphi_x(x, t) = -\lambda_1\}| - \frac{\lambda_2}{\lambda_1 + \lambda_2} |Q| \right| < \epsilon, \\ \left| |\{(x, t) \in Q \mid \varphi_x(x, t) = \lambda_2\}| - \frac{\lambda_1}{\lambda_1 + \lambda_2} |Q| \right| < \epsilon, \end{cases}$
- (d) $\psi_x = \varphi$ in Q , and
- (e) $\int_{x_1}^{x_2} \varphi(x, t) dx = 0$ for all $t_1 < t < t_2$.

Proof. Let τ and ν be sufficiently small positive numbers to be specified below. We first choose a function $h_\tau \in C_c^\infty(t_1, t_2)$ so that

$$0 \leq h_\tau \leq 1 \text{ in } (t_1, t_2) \text{ and } h_\tau = 1 \text{ on } [t_1 + \tau, t_2 - \tau].$$

We also choose a (highly oscillating) function $f_\nu \in C_c^\infty(x_1, x_2)$ such that

$$\begin{cases} -\lambda_1 \leq f'_\nu \leq \lambda_2 \text{ in } (x_1, x_2), \\ \left| |\{x \in (x_1, x_2) \mid f'_\nu(x) = -\lambda_1\}| - \frac{\lambda_2}{\lambda_1 + \lambda_2} (x_2 - x_1) \right| < \nu, \\ \left| |\{x \in (x_1, x_2) \mid f'_\nu(x) = \lambda_2\}| - \frac{\lambda_1}{\lambda_1 + \lambda_2} (x_2 - x_1) \right| < \nu, \\ \int_{x_1}^{x_2} f_\nu(x) dx = 0, \\ \|f_\nu\|_{L^\infty(x_1, x_2)} < \nu. \end{cases}$$

Define $\varphi(x, t) = \varphi_{\tau, \nu}(x, t) = h_\tau(t) f_\nu(x)$ for $(x, t) \in Q$; then $\varphi \in C_c^\infty(Q)$ and $\int_{x_1}^{x_2} \varphi(x, t) dx = 0$ for all $t_1 < t < t_2$; so (e) holds. Note also that $\|\varphi\|_{L^\infty(Q)} < \nu$, $\|\varphi_t\|_{L^\infty(Q)} < \nu \|h'_\tau\|_{L^\infty(t_1, t_2)}$, $-\lambda_1 \leq \varphi_x \leq \lambda_2$ in Q ,

$$\begin{aligned} & \left| |\{(x, t) \in Q \mid \varphi_x(x, t) = -\lambda_1\}| - \frac{\lambda_2}{\lambda_1 + \lambda_2} |Q| \right| \\ & \leq \left| |\{(x, t) \in Q \mid \varphi_x(x, t) = -\lambda_1\}| - |\{x \in (x_1, x_2) \mid f'_\nu(x) = -\lambda_1\}| \times (t_1, t_2)| \right| \\ & \quad + \left| |\{x \in (x_1, x_2) \mid f'_\nu(x) = -\lambda_1\}| \times (t_1, t_2) - \frac{\lambda_2}{\lambda_1 + \lambda_2} |Q| \right| \\ & < 2\tau(x_2 - x_1) + \nu(t_2 - t_1), \text{ and} \end{aligned}$$

$$\left| |\{(x, t) \in Q \mid \varphi_x(x, t) = \lambda_2\}| - \frac{\lambda_1}{\lambda_1 + \lambda_2} |Q| \right| < 2\tau(x_2 - x_1) + \nu(t_2 - t_1);$$

here, we have (b).

Next, define $\psi(x, t) = \psi_{\tau, \nu}(x, t) = \int_{x_1}^x \varphi(y, t) dy$ for $(x, t) \in Q$. Then it is easily checked that $\psi \in C_c^\infty(Q)$ (by (e)), $\|\psi\|_{L^\infty(Q)} < \nu(x_2 - x_1)$, $\psi_x = \varphi$ in

Q , and $\|\psi_t\|_{L^\infty(Q)} < \nu(x_2 - x_1)\|h'_\tau\|_{L^\infty(t_1, t_2)}$; thus $\omega := (\varphi, \psi) \in C_c^\infty(Q; \mathbb{R}^2)$, and (d) is satisfied.

In the above context, for any given $\epsilon > 0$, we first choose a $\tau > 0$ so small that $2\tau(x_2 - x_1) < \epsilon/2$. We then choose a $\nu > 0$ in such a way that

$$\nu(x_2 - x_1 + 1) < \epsilon, \quad \nu\|h'_\tau\|_{L^\infty(t_1, t_2)} < \epsilon, \quad \nu(t_2 - t_1) < \epsilon/2, \quad \text{and}$$

$$\nu(x_2 - x_1)\|h'_\tau\|_{L^\infty(t_1, t_2)} < \epsilon.$$

Then (a) and (c) are also fulfilled. \square

7. PROOF OF DENSITY LEMMA

As the final task of the paper, we provide a long proof of the density lemma, Lemma 4.1.

Proof of Lemma 4.1. Let $E = \Omega_T^2$. Fix a $\delta > 0$, and let $w = (u, v) \in \mathcal{A}$; so

$$(7.1) \quad \begin{cases} w \in C^{2,1}(\bar{\Omega}_T) \times C^{3,1}(\bar{\Omega}_T), \\ w = w^* \text{ in } \Omega_T \setminus \bar{\Omega}_T^w \text{ for some regular open set } \Omega_T^w \subset \subset \Omega_T^2, \\ \nabla w(x, t) \in U((x, t); u) \quad \forall (x, t) \in \Omega_T^2, \\ \|u - u^*\|_\infty < \epsilon/2, \quad \|u_t - u_t^*\|_\infty < \epsilon/2, \\ \mathcal{P}_u = \mathcal{P}_{u^*} \text{ in } \Omega_T \setminus \bar{\Omega}_T^w, \text{ and } |\Gamma_w^E - \Gamma_{w^*}^E| < \epsilon/2. \end{cases}$$

Let $\eta > 0$. Our goal is to construct a function $w_\eta = (u_\eta, v_\eta) \in \mathcal{A}_\delta$ with $\|w_\eta - w\|_\infty < \eta$, that is, a function $w_\eta \in C^{2,1}(\bar{\Omega}_T) \times C^{3,1}(\bar{\Omega}_T)$ satisfying

$$(7.2) \quad \begin{cases} w_\eta = w^* \text{ in } \Omega_T \setminus \bar{\Omega}_T^{w_\eta} \text{ for some regular open set } \Omega_T^{w_\eta} \subset \subset \Omega_T^2, \\ \nabla w_\eta(x, t) \in U((x, t); u_\eta) \quad \forall (x, t) \in \Omega_T^2, \\ \|u_\eta - u^*\|_\infty < \epsilon/2, \quad \|(u_\eta)_t - u_t^*\|_\infty < \epsilon/2, \\ \mathcal{P}_{u_\eta} = \mathcal{P}_{u^*} \text{ in } \Omega_T \setminus \bar{\Omega}_T^{w_\eta}, \quad |\Gamma_{w_\eta}^E - \Gamma_{w^*}^E| < \epsilon/2, \\ \int_{\Omega_T^2} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \leq \delta|\Omega_T^2|, \text{ and} \\ \|w_\eta - w\|_\infty < \eta. \end{cases}$$

The construction is so long that we divide it into several steps.

Step 1. First, we choose a regular open set $G \subset \subset \Omega_T^2 \setminus \partial\Omega_T^w$ so that

$$(7.3) \quad \int_{\Omega_T^2 \setminus \bar{G}} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dxdt \leq \frac{\delta}{k}|\Omega_T^2|,$$

where $k \in \mathbb{N}$ is to be specified later.

By (7.1), we have

$$(u_x, v_t - bu - \mathcal{P}_u - F) \in U', \quad |u_t| < m^* \quad \text{on } \bar{G};$$

so

$$(7.4) \quad \begin{aligned} d' &:= \min_{\bar{G}} \text{dist}((u_x, v_t - bu - \mathcal{P}_u - F), \partial U') > 0, \\ m' &:= m^* - \max_{\bar{G}} |u_t| > 0. \end{aligned}$$

By (7.1), we also have

$$d'' := (\epsilon/2 - |\Gamma_w^E - \Gamma_{w*}^E|)/2 > 0.$$

By the uniform continuity of s_r^\pm for $r \in [\sigma(s_2), \sigma(s_1)]$, we can choose a $\kappa > 0$ such that

$$(7.5) \quad |s_a^\pm - s_b^\pm| \leq d''/l \text{ whenever } a, b \in [\sigma(s_2), \sigma(s_1)] \text{ and } |a - b| \leq \kappa,$$

where $l \in \mathbb{N}$ will be chosen later.

Next, we choose finitely many disjoint open squares $Q_1, \dots, Q_N \subset G$, parallel to the axes, such that

$$(7.6) \quad \int_{G \setminus (\cup_{i=1}^N \bar{Q}_i)} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dx dt \leq \frac{\delta}{k} |\Omega_T^2|.$$

Dividing these squares $Q_1, \dots, Q_N \subset G$ into disjoint open sub-squares if necessary, we can assume that

$$(7.7) \quad \begin{aligned} & |(u_x(x_1, t_1), v_t(x_1, t_1) - b(x_1, t_1)u(x_1, t_1) - \mathcal{P}_u(x_1, t_1) - F(x_1, t_1)) \\ & - (u_x(x_2, t_2), v_t(x_2, t_2) - b(x_2, t_2)u(x_2, t_2) - \mathcal{P}_u(x_2, t_2) - F(x_2, t_2))| \\ & \leq \min \left\{ \frac{\delta}{4k}, \frac{d'}{4}, \frac{\kappa}{2} \right\} \end{aligned}$$

and

$$(7.8) \quad |u_t(x_1, t_1) - u_t(x_2, t_2)| \leq \frac{m'}{4}$$

whenever $(x_1, t_1), (x_2, t_2) \in \bar{Q}_i$ and $i \in \{1, \dots, N\}$.

Step 2. Fix an index $i \in \mathcal{I} := \{1, \dots, N\}$, and let us denote by (x_i, t_i) the center of the square Q_i . We write

$$\begin{aligned} (s_i, \gamma_i) &= (u_x(x_i, t_i), v_t(x_i, t_i) - b(x_i, t_i)u(x_i, t_i) - \mathcal{P}_u(x_i, t_i) - F(x_i, t_i)), \\ c_i &= u_t(x_i, t_i); \end{aligned}$$

then $(s_i, \gamma_i) \in U'$ and $|c_i| \leq m^* - m'$.

We now split the index set \mathcal{I} into

$$\mathcal{I}_1 := \{i \in \mathcal{I} \mid \text{dist}((s_i, \gamma_i), K') \leq \delta/k\}, \quad \mathcal{I}_2 := \mathcal{I} \setminus \mathcal{I}_1.$$

Let $i \in \mathcal{I}_1$. Choose a point $\bar{s}_i \in [s_{r_1}^-, s_{r_2}^-] \cup [s_{r_1}^+, s_{r_2}^+]$ so that

$$(7.9) \quad |(s_i, \gamma_i) - (\bar{s}_i, \sigma(\bar{s}_i))| \leq \frac{\delta}{k}.$$

Let $(x, t) \in Q_i$; then by (7.7) and (7.9), we have

$$\begin{aligned} & \left| \begin{pmatrix} u_x(x, t) & u_t(x, t) \\ v_x(x, t) & v_t(x, t) \end{pmatrix} - \begin{pmatrix} \bar{s}_i & u_t(x, t) \\ u(x, t) & b(x, t)u(x, t) + \mathcal{P}_u(x, t) + F(x, t) + \sigma(\bar{s}_i) \end{pmatrix} \right| \\ &= |(u_x(x, t) - \bar{s}_i, v_t(x, t) - b(x, t)u(x, t) - \mathcal{P}_u(x, t) - F(x, t) - \sigma(\bar{s}_i))| \\ &\leq |(u_x(x, t), v_t(x, t) - b(x, t)u(x, t) - \mathcal{P}_u(x, t) - F(x, t)) - (s_i, \gamma_i)| \\ &\quad + |(s_i, \gamma_i) - (\bar{s}_i, \sigma(\bar{s}_i))| \leq \frac{5\delta}{4k}, \end{aligned}$$

where

$$\begin{pmatrix} \bar{s}_i & u_t(x, t) \\ u(x, t) & b(x, t)u(x, t) + \mathcal{P}_u(x, t) + F(x, t) + \sigma(\bar{s}_i) \end{pmatrix} \in K((x, t); u).$$

So

$$\int_{Q_i} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dxdt \leq \frac{5\delta}{4k} |Q_i|.$$

We thus have

$$(7.10) \quad \int_{\cup_{i \in \mathcal{I}_1} Q_i} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dxdt \leq \frac{5\delta}{4k} |\Omega_T^2|.$$

Step 3. Fix an index $i \in \mathcal{I}_2$. Since $(s_i, \gamma_i) \in U'$ and $\text{dist}((s_i, \gamma_i), K') > \delta/k$, we can choose two numbers $\lambda_{i,1} > 0$ and $\lambda_{i,2} > 0$ so that

$$(7.11) \quad \begin{cases} (s_i - \lambda_{i,1}, \gamma_i), (s_i + \lambda_{i,2}, \gamma_i) \in U', \\ \text{dist}((s_i - \lambda_{i,1}, \gamma_i), K') = \text{dist}((s_i + \lambda_{i,2}, \gamma_i), K') = \frac{\delta}{k}, \\ \text{dist}((s_i + \mu, \gamma_i), K') > \frac{\delta}{k} \quad \forall \mu \in (-\lambda_{i,1}, \lambda_{i,2}). \end{cases}$$

For a given $\epsilon_i > 0$ to be specified later, we can apply Lemma 6.1 to obtain a function $\omega_i = (\varphi_i, \psi_i) \in C_c^\infty(Q_i; \mathbb{R}^2)$ such that

- (a) $\|\omega_i\|_{L^\infty(Q_i)} < \epsilon_i$, $\|(\varphi_i)_t\|_{L^\infty(Q_i)} < \epsilon_i$, $\|(\psi_i)_t\|_{L^\infty(Q_i)} < \epsilon_i$,
- (b) $-\lambda_{i,1} \leq (\varphi_i)_x \leq \lambda_{i,2}$ in Q_i ,
- (c) $\begin{cases} |\{(x, t) \in Q_i \mid (\varphi_i)_x(x, t) = -\lambda_{i,1}\}| - \frac{\lambda_{i,2}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| < \epsilon_i, \\ |\{(x, t) \in Q_i \mid (\varphi_i)_x(x, t) = \lambda_{i,2}\}| - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| < \epsilon_i, \end{cases}$
- (d) $(\psi_i)_x = \varphi_i$ in Q_i , and
- (e) $\int_{x_{i,1}}^{x_{i,2}} \varphi_i(x, t) \, dx = 0$ for all $t_{i,1} < t < t_{i,2}$,

where $Q_i = (x_{i,1}, x_{i,2}) \times (t_{i,1}, t_{i,2})$. We then define

$$(7.12) \quad w_\eta = (u_\eta, v_\eta) = w + \sum_{i \in \mathcal{I}_2} \omega_i \chi_{Q_i} \quad \text{in } \Omega_T.$$

Step 4. To finish the proof, we now show that upon choosing suitable numbers $l, k \in \mathbb{N}$ and $\epsilon_i > 0$ ($i \in \mathcal{I}_2$), the function $w_\eta = (u_\eta, v_\eta)$ defined in (7.12) satisfies the required properties in (7.2). As this step consists of many arguments to verify, we separate it into several substeps.

Substep 4-1. We begin with relatively easy parts to show.

From (7.1) and (7.12), we have

$$(7.13) \quad w_\eta \in C^{2,1}(\bar{\Omega}_T) \times C^{3,1}(\bar{\Omega}_T).$$

Set $\Omega_T^{w_\eta} = G \cup \Omega_T^w$; then $\Omega_T^{w_\eta} \subset \subset \Omega_T^2$ is a regular open set. From (7.1) and (7.12), we also have

$$(7.14) \quad w_\eta = w = w^* \quad \text{in } \Omega_T \setminus \bar{\Omega}_T^{w_\eta}.$$

Next, we choose

$$(7.15) \quad \epsilon_i < \min \left\{ \eta, \frac{\epsilon}{2} - \|u - u^*\|_\infty, \frac{\epsilon}{2} - \|u_t - u_t^*\|_\infty \right\} \quad \text{for all } i \in \mathcal{I}_2;$$

then from (a) and (7.12) we have

$$(7.16) \quad \begin{cases} \|w_\eta - w\|_\infty < \eta, \\ \|u_\eta - u^*\|_\infty < \epsilon/2, \\ \|(u_\eta)_t - u_t^*\|_\infty < \epsilon/2. \end{cases}$$

Note that by (1.5), (e), (7.1) and (7.12), for all $(x, t) \in \Omega_T \setminus \bar{\Omega}_T^{w_\eta}$,

$$\begin{aligned} \mathcal{P}_{u_\eta}(x, t) &= d(t) \int_0^x u_\eta(y, t) dy \\ &= d(t) \int_0^x u(y, t) dy + d(t) \sum_{i \in \mathcal{I}_2} \int_0^x \varphi_i(y, t) \chi_{Q_i}(y, t) dy \\ &= \mathcal{P}_u(x, t) = \mathcal{P}_{u^*}(x, t); \end{aligned}$$

that is,

$$(7.17) \quad \mathcal{P}_{u_\eta} = \mathcal{P}_{u^*} \text{ in } \Omega_T \setminus \bar{\Omega}_T^{w_\eta}.$$

Substep 4-2. In this substep, we show that

$$(7.18) \quad \nabla w_\eta(x, t) \in U((x, t); u_\eta) \quad \forall (x, t) \in \Omega_T^2.$$

Note from (7.8), (7.12) and (a) that

$$\begin{aligned} |(u_\eta)_t(x, t)| &= |u_t(x, t) + (\varphi_i)_t(x, t)| \leq |u_t(x, t) - c_i| + |c_i| + |(\varphi_i)_t(x, t)| \\ &\leq \frac{m'}{4} + m^* - m' + \epsilon_i < m^* - \frac{m'}{2} \quad \forall (x, t) \in Q_i, \quad \forall i \in \mathcal{I}_2, \end{aligned}$$

where we let

$$(7.19) \quad \epsilon_i < m'/4 \text{ for each } i \in \mathcal{I}_2.$$

This implies that

$$(7.20) \quad |(u_\eta)_t(x, t)| < m^* \quad \forall (x, t) \in \Omega_T^2.$$

From (7.1), we have $v_x = u$ in Ω_T^2 . Thus by (7.12) and (d), we have

$$\begin{aligned} (v_\eta)_x(x, t) &= v_x(x, t) + (\psi_i)_x(x, t) \\ &= u(x, t) + \varphi_i(x, t) = u_\eta(x, t) \quad \forall (x, t) \in Q_i, \end{aligned}$$

where $i \in \mathcal{I}_2$; that is,

$$(7.21) \quad (v_\eta)_x(x, t) = u_\eta(x, t) \quad \forall (x, t) \in \Omega_T^2.$$

Let $i \in \mathcal{I}_2$. From (7.4) and (7.11), we see that

$$(7.22) \quad \text{dist}([(s_i - \lambda_{i,1}, \gamma_i), (s_i + \lambda_{i,2}, \gamma_i)], \partial U') \geq \min \left\{ \frac{\delta}{k}, d' \right\}.$$

Let $(x, t) \in Q_i$. Then by (a), (e) and (7.7),

$$\begin{aligned}
& \left| \left((u_\eta)_x(x, t), (v_\eta)_t(x, t) - b(x, t)u_\eta(x, t) - \mathcal{P}_{u_\eta}(x, t) - F(x, t) \right) \right. \\
& \quad \left. - (s_i + (\varphi_i)_x(x, t), \gamma_i) \right| \\
& \leq \left| \left(u_x(x, t), v_t(x, t) - b(x, t)u(x, t) - \mathcal{P}_u(x, t) - F(x, t) \right) - (s_i, \gamma_i) \right| \\
& \quad + \left| (\psi_i)_t(x, t) - b(x, t)\varphi_i(x, t) - d(t) \int_{x_{i,1}}^x \varphi_i(y, t) dy \right| \\
& \leq \min \left\{ \frac{\delta}{4k}, \frac{d'}{4} \right\} + \epsilon_i(1 + \|b\|_\infty + \|d\|_{L^\infty(0,T)}) < \min \left\{ \frac{\delta}{2k}, \frac{d'}{2} \right\},
\end{aligned}$$

where we let

$$(7.23) \quad \epsilon_i < (1 + \|b\|_\infty + \|d\|_{L^\infty(0,T)})^{-1} \min \left\{ \frac{\delta}{4k}, \frac{d'}{4} \right\}.$$

By (b), we have $-\lambda_{i,1} \leq (\varphi_i)_x(x, t) \leq \lambda_{i,2}$, and so it follows from (7.22) and the previous inequality that

$$((u_\eta)_x(x, t), (v_\eta)_t(x, t) - b(x, t)u_\eta(x, t) - \mathcal{P}_{u_\eta}(x, t) - F(x, t)) \in U'.$$

Adopting (7.23) for all $i \in \mathcal{I}_2$, this inclusion holds for all $(x, t) \in \Omega_T^2$.

This inclusion together with (7.20) and (7.21) implies inclusion (7.18).

Substep 4-3. Here, we prove that

$$(7.24) \quad |\Gamma_{w_\eta}^E - \Gamma_{w^*}^E| < \epsilon/2.$$

Observe from (2.6) that we have

$$\begin{aligned}
\Gamma_{w_\eta}^E - \Gamma_w^E &= \frac{1}{|\Omega_T^2|} \int_{\Omega_T^2} (Z_{w_\eta}(x, t) - Z_w(x, t)) dx dt \\
&= \frac{1}{|\Omega_T^2|} \sum_{i \in \mathcal{I}_2} \int_{Q_i} (Z_{w+\omega_i}(x, t) - Z_w(x, t)) dx dt \\
&= \frac{1}{|\Omega_T^2|} \sum_{i \in \mathcal{I}_2} \left(\int_{Q_{i,1}} + \int_{Q_{i,2}^+} + \int_{Q_{i,2}^-} \right) (Z_{w+\omega_i} - Z_w),
\end{aligned}$$

where

$$\begin{aligned}
Q_{i,1} &:= \{(x, t) \in Q_i \mid (\varphi_i)_x(x, t) \notin \{-\lambda_{i,1}, \lambda_{i,2}\}\}, \\
Q_{i,2}^+ &:= \{(x, t) \in Q_i \mid (\varphi_i)_x(x, t) = \lambda_{i,2}\}, \\
Q_{i,2}^- &:= \{(x, t) \in Q_i \mid (\varphi_i)_x(x, t) = -\lambda_{i,1}\}.
\end{aligned}$$

For the second equality above, we have used the facts from (e) that $\mathcal{P}_{u_\eta}(x, t) = \mathcal{P}_u(x, t)$ for all $(x, t) \notin \cup_{i \in \mathcal{I}_2} Q_i$ and that $\mathcal{P}_{u_\eta}(x, t) = \mathcal{P}_{u+\varphi_i}(x, t)$ for all $(x, t) \in Q_i$ and $i \in \mathcal{I}_2$; here, ω_i is defined to be zero outside its compact support.

The trivial part to estimate here follows from (c) as

$$(7.25) \quad \left| \int_{Q_{i,1}} (Z_{w+\omega_i} - Z_w) \right| \leq 2|Q_{i,1}| < 4\epsilon_i \quad (i \in \mathcal{I}_2).$$

Next, let $i \in \mathcal{I}_2$ and $(x, t) \in Q_{i,2}^-$. Then from (7.7), we have at the point (x, t) that

$$\begin{aligned}
 |(u + \varphi_i)_x - S_{w+\omega_i}^-| &= |(u + \varphi_i)_x - s_{(v+\psi_i)_t-b(u+\varphi_i)-\mathcal{P}_{u+\varphi_i}-F}^-| \\
 &= |u_x - \lambda_{i,1} - s_{(v+\psi_i)_t-b(u+\varphi_i)-\mathcal{P}_{u+\varphi_i}-F}^-| \\
 (7.26) \quad &\leq |u_x - s_i| + |s_i - \lambda_{i,1} - s_{\gamma_i}^-| \\
 &\quad + |s_{\gamma_i}^- - s_{(v+\psi_i)_t-b(u+\varphi_i)-\mathcal{P}_{u+\varphi_i}-F}^-| \\
 &\leq \frac{\delta}{4k} + |s_i - \lambda_{i,1} - s_{\gamma_i}^-| + |s_{\gamma_i}^- - s_{(v+\psi_i)_t-b(u+\varphi_i)-\mathcal{P}_{u+\varphi_i}-F}^-|.
 \end{aligned}$$

By (7.11), we have $|(s_i - \lambda_{i,1}, \gamma_i) - (\bar{s}, \sigma(\bar{s}))| = \delta/k$ for some $\bar{s} \in [s_{\gamma_i}^-, s_i - \lambda_{i,1}]$. Thus, we have from (7.5) that

$$(7.27) \quad |s_i - \lambda_{i,1} - s_{\gamma_i}^-| \leq |s_i - \lambda_{i,1} - \bar{s}| + |\bar{s} - s_{\gamma_i}^-| \leq \frac{\delta}{k} + \frac{d''}{l},$$

where we let $k \in \mathbb{N}$ satisfy

$$(7.28) \quad \frac{\delta}{k} \leq \kappa.$$

Likewise, we also have

$$(7.29) \quad |s_i + \lambda_{i,2} - s_{\gamma_i}^+| \leq \frac{\delta}{k} + \frac{d''}{l}.$$

Also, from (7.7) and (a), we have

$$\begin{aligned}
 &|\gamma_i - ((v + \psi_i)_t - b(u + \varphi_i) - \mathcal{P}_{u+\varphi_i} - F)| \\
 &\leq |\gamma_i - (v_t - bu - \mathcal{P}_u - F)| \\
 &\quad + |(\psi_i)_t| + |b||\varphi_i| + |d(t)| \left| \int_0^x \varphi_i(y, t) dy \right| \\
 &\leq \frac{\kappa}{2} + \epsilon_i(1 + \|b\|_\infty + \|d\|_{L^\infty(0,T)}) \leq \kappa,
 \end{aligned}$$

where we let

$$(7.30) \quad \epsilon_i \leq 2^{-1}(1 + \|b\|_\infty + \|d\|_{L^\infty(0,T)})^{-1}\kappa.$$

With the help of (7.5), this implies that

$$(7.31) \quad |s_{\gamma_i}^- - S_{w+\omega_i}^-| = |s_{\gamma_i}^- - s_{(v+\psi_i)_t-b(u+\varphi_i)-\mathcal{P}_{u+\varphi_i}-F}^-| \leq \frac{d''}{l}.$$

Thus (7.26), (7.27) and (7.31) yield that

$$|(u + \varphi_i)_x - s_{(v+\psi_i)_t-b(u+\varphi_i)-\mathcal{P}_{u+\varphi_i}-F}^-| \leq \frac{5\delta}{4k} + \frac{2d''}{l} \leq \frac{3d''}{l}$$

if we choose $k \in \mathbb{N}$ so large that

$$(7.32) \quad \frac{5\delta}{4k} \leq \frac{d''}{l}.$$

Since $S_{w+\omega_i}^+ - S_{w+\omega_i}^- \geq s_{r_1}^+ - s_{r_2}^-$, we thus have

$$Z_{w+\omega_i}(x, t) \leq \frac{3d''}{l(s_{r_1}^+ - s_{r_2}^-)} \quad \forall (x, t) \in Q_{i,2}^-.$$

Hence

$$(7.33) \quad \left| \int_{Q_{i,2}^-} Z_{w+\omega_i} \right| \leq \frac{3d''}{l(s_{r_1}^+ - s_{r_2}^-)} |Q_{i,2}^-|.$$

Let $i \in \mathcal{I}_2$ and $Q_{i,2} := Q_{i,2}^+ \cup Q_{i,2}^-$. We now estimate the quantity

$$\left| \int_{Q_{i,2}^+} Z_{w+\omega_i} - \int_{Q_{i,2}} Z_w \right|.$$

First, we consider

$$\begin{aligned} & \left| \int_{Q_{i,2}^+} Z_{w+\omega_i}(x, t) dx dt - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| \\ & \leq \left| \int_{Q_{i,2}^+} \left(\frac{u_x + (\varphi_i)_x - S_{w+\omega_i}^-}{S_{w+\omega_i}^+ - S_{w+\omega_i}^-} - \frac{s_i + \lambda_{i,2} - s_{\gamma_i}^-}{S_{w+\omega_i}^+ - S_{w+\omega_i}^-} \right) dx dt \right| \\ & \quad + \left| \int_{Q_{i,2}^+} \left(\frac{s_i + \lambda_{i,2} - s_{\gamma_i}^-}{S_{w+\omega_i}^+ - S_{w+\omega_i}^-} - \frac{s_i + \lambda_{i,2} - s_{\gamma_i}^-}{s_{\gamma_i}^+ - s_{\gamma_i}^-} \right) dx dt \right| \\ & \quad + \left| \frac{s_i + \lambda_{i,2} - s_{\gamma_i}^-}{s_{\gamma_i}^+ - s_{\gamma_i}^-} |Q_{i,2}^+| - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| =: I_1 + I_2 + I_3. \end{aligned}$$

Here, let $\epsilon_i > 0$ satisfy (7.30); then as in (7.31), we have

$$(7.34) \quad |S_{w+\omega_i}^\pm - s_{\gamma_i}^\pm| \leq \frac{d''}{l} \quad \text{in } Q_{i,2}^+.$$

Using this, (7.7) and the fact that $(\varphi_i)_x = \lambda_{i,2}$ in $Q_{i,2}^+$, we deduce that

$$I_1 \leq (s_{r_1}^+ - s_{r_2}^-)^{-1} \left(\frac{\delta}{4k} + \frac{d''}{l} \right) |Q_{i,2}^+| \leq \frac{6d''}{5l(s_{r_1}^+ - s_{r_2}^-)} |Q_{i,2}^+|,$$

where we let $k \in \mathbb{N}$ fulfill (7.32). Having the common denominator in the integrand, we have from (7.34) that

$$I_2 \leq (s_i + \lambda_{i,2} - s_{\gamma_i}^-)(s_{r_1}^+ - s_{r_2}^-)^{-2} \frac{2d''}{l} |Q_{i,2}^+| \leq \frac{2d''(s_{r_2}^+ - s_{r_1}^-)}{l(s_{r_1}^+ - s_{r_2}^-)^2} |Q_{i,2}^+|.$$

Note from (7.29) and (c) that

$$\begin{aligned} I_3 & \leq \left| \frac{s_i + \lambda_{i,2} - s_{\gamma_i}^-}{s_{\gamma_i}^+ - s_{\gamma_i}^-} |Q_{i,2}^+| - |Q_{i,2}^+| \right| + \left| |Q_{i,2}^+| - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| \\ & \leq \left(\frac{\delta}{k} + \frac{d''}{l} \right) (s_{\gamma_i}^+ - s_{\gamma_i}^-)^{-1} |Q_{i,2}^+| + \epsilon_i \leq \frac{9d''}{5l(s_{r_1}^+ - s_{r_2}^-)} |Q_{i,2}^+| + \epsilon_i, \end{aligned}$$

where we let $k \in \mathbb{N}$ also satisfy (7.28). Combining the estimates on I_1 , I_2 and I_3 , we obtain

$$(7.35) \quad \left| \int_{Q_{i,2}^+} Z_{w+\omega_i}(x,t) dxdt - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| \leq \left(\frac{3}{s_{r_1}^+ - s_{r_2}^-} + \frac{2(s_{r_2}^+ - s_{r_1}^-)}{(s_{r_1}^+ - s_{r_2}^-)^2} \right) \frac{d''}{l} |Q_{i,2}^+| + \epsilon_i.$$

Second, we handle

$$\begin{aligned} & \left| \int_{Q_{i,2}} Z_w(x,t) dxdt - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| \\ & \leq \left| \int_{Q_{i,2}} \left(\frac{u_x - S_w^-}{S_w^+ - S_w^-} - \frac{\lambda_{i,1}}{S_w^+ - S_w^-} \right) dxdt \right| \\ & \quad + \left| \int_{Q_{i,2}} \frac{\lambda_{i,1}}{S_w^+ - S_w^-} dxdt - \frac{\lambda_{i,1}}{s_{\gamma_i}^+ - s_{\gamma_i}^-} |Q_{i,2}| \right| \\ & \quad + \left| \frac{\lambda_{i,1}}{s_{\gamma_i}^+ - s_{\gamma_i}^-} |Q_{i,2}| - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| =: J_1 + J_2 + J_3. \end{aligned}$$

To estimate J_1 , we first note from (7.5), (7.7) and (7.27) that for $(x,t) \in Q_{i,2}$,

$$\begin{aligned} |u_x(x,t) - S_w^-(x,t) - \lambda_{i,1}| & \leq |u_x(x,t) - s_i| + |s_{\gamma_i}^- - S_w^-(x,t)| + |s_i - \lambda_{i,1} - s_{\gamma_i}^-| \\ & \leq \frac{\delta}{4k} + \frac{d''}{l} + \left(\frac{\delta}{k} + \frac{d''}{l} \right) \leq \frac{3d''}{l}, \end{aligned}$$

where we let $k \in \mathbb{N}$ satisfy (7.28) and (7.32). From this, we get

$$J_1 \leq \frac{3d''}{l(s_{r_1}^+ - s_{r_2}^-)} |Q_{i,2}|.$$

From (7.5) and (7.7), we have

$$J_2 \leq \frac{2\lambda_{i,1}d''}{l(s_{r_1}^+ - s_{r_2}^-)^2} |Q_{i,2}| \leq \frac{2d''(s_{r_2}^+ - s_{r_1}^-)}{l(s_{r_1}^+ - s_{r_2}^-)^2} |Q_{i,2}|.$$

To estimate J_3 , we observe from (7.27) and (7.29) that

$$\begin{aligned} |\lambda_{i,1} + \lambda_{i,2} - (s_{\gamma_i}^+ - s_{\gamma_i}^-)| & = |s_i + \lambda_{i,2} - s_{\gamma_i}^+ - (s_i - \lambda_{i,1} - s_{\gamma_i}^-)| \\ & \leq \frac{2\delta}{k} + \frac{2d''}{l} \leq \frac{18d''}{5l}. \end{aligned}$$

Note from this and (c) that

$$\begin{aligned} J_3 & \leq \left| \frac{\lambda_{i,1}}{s_{\gamma_i}^+ - s_{\gamma_i}^-} |Q_{i,2}| - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_{i,2}| \right| + \left| \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_{i,2}| - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| \\ & \leq \frac{18d''(s_{r_2}^+ - s_{r_1}^-)}{5l(s_{r_1}^+ - s_{r_2}^-)^2} |Q_{i,2}| + 2\epsilon_i. \end{aligned}$$

By the estimates on J_1 , J_2 and J_3 , we now have

$$\begin{aligned} \left| \int_{Q_{i,2}} Z_w(x, t) dx dt - \frac{\lambda_{i,1}}{\lambda_{i,1} + \lambda_{i,2}} |Q_i| \right| \\ \leq \left(\frac{3}{s_{r_1}^+ - s_{r_2}^-} + \frac{28(s_{r_2}^+ - s_{r_1}^-)}{5(s_{r_1}^+ - s_{r_2}^-)^2} \right) \frac{d''}{l} |Q_{i,2}| + 2\epsilon_i. \end{aligned}$$

Combining this estimate with (7.35), we have

$$\begin{aligned} (7.36) \quad \left| \int_{Q_{i,2}^+} Z_{w+\omega_i} - \int_{Q_{i,2}} Z_w \right| \\ \leq \left(\frac{6}{s_{r_1}^+ - s_{r_2}^-} + \frac{38(s_{r_2}^+ - s_{r_1}^-)}{5(s_{r_1}^+ - s_{r_2}^-)^2} \right) \frac{d''}{l} |Q_{i,2}| + 3\epsilon_i. \end{aligned}$$

Thanks to the estimates (7.25), (7.33) and (7.36), it follows that for all $i \in \mathcal{I}_2$,

$$\begin{aligned} \left| \left(\int_{Q_{i,1}} + \int_{Q_{i,2}^+} + \int_{Q_{i,2}^-} \right) (Z_{w+\omega_i} - Z_w) \right| \\ \leq \left(\frac{9}{s_{r_1}^+ - s_{r_2}^-} + \frac{38(s_{r_2}^+ - s_{r_1}^-)}{5(s_{r_1}^+ - s_{r_2}^-)^2} \right) \frac{d''}{l} |Q_{i,2}| + 7\epsilon_i \\ \leq \frac{17d''(s_{r_2}^+ - s_{r_1}^-)}{l(s_{r_1}^+ - s_{r_2}^-)^2} |Q_i| + 7\epsilon_i. \end{aligned}$$

Summing this over the indices $i \in \mathcal{I}_2$, we then have

$$|\Gamma_{w_\eta}^E - \Gamma_w^E| \leq \frac{17d''(s_{r_2}^+ - s_{r_1}^-)}{l(s_{r_1}^+ - s_{r_2}^-)^2} + \frac{7}{|\Omega_T^2|} \sum_{i \in \mathcal{I}_2} \epsilon_i < d''$$

if $l \in \mathbb{N}$ is taken so large that

$$(7.37) \quad \frac{17(s_{r_2}^+ - s_{r_1}^-)}{l(s_{r_1}^+ - s_{r_2}^-)^2} < \frac{1}{2},$$

$k \in \mathbb{N}$ satisfies (7.28) and (7.32), and the numbers $\epsilon_i > 0$ ($i \in \mathcal{I}_2$) fulfill (7.30) and

$$\epsilon_i < \frac{d''}{14N} |\Omega_T^2|.$$

Under these choices of numbers l , k and ϵ_i ($i \in \mathcal{I}_2$), we finally have from (7.1) and the definition of d'' that

$$\begin{aligned} |\Gamma_{w_\eta}^E - \Gamma_{w^*}^E| &\leq |\Gamma_{w_\eta}^E - \Gamma_w^E| + |\Gamma_w^E - \Gamma_{w^*}^E| \\ &< d'' + |\Gamma_w^E - \Gamma_{w^*}^E| = \frac{\epsilon}{4} + \frac{|\Gamma_w^E - \Gamma_{w^*}^E|}{2} < \frac{\epsilon}{2}; \end{aligned}$$

hence, our goal (7.24) for this substep is indeed achieved.

Substep 4-4. We now prove

$$(7.38) \quad \int_{\Omega_T^2} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \leq \delta |\Omega_T^2|.$$

Let $i \in \mathcal{I}_2$. By (c), we have $|Q_{i,1}| < 2\epsilon_i$. Let $(x, t) \in Q_{i,1}$ and $s \in [s_{r_1}^-, s_{r_2}^-] \cup [s_{r_1}^+, s_{r_2}^+]$; then by (7.18), we have, at the point (x, t) ,

$$\begin{aligned} & \left| \begin{pmatrix} (u_\eta)_x & (u_\eta)_t \\ (v_\eta)_x & (v_\eta)_t \end{pmatrix} - \begin{pmatrix} s & (u_\eta)_t \\ u_\eta & bu_\eta + \mathcal{P}_{u_\eta} + F + \sigma(s) \end{pmatrix} \right| \\ &= |((u_\eta)_x, (v_\eta)_t - bu_\eta - \mathcal{P}_{u_\eta} - F) - (s, \sigma(s))| \\ &\leq \text{diam}(U'). \end{aligned}$$

Thus, we have

$$\int_{Q_{i,1}} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \leq 2\epsilon_i \text{diam}(U') \leq \frac{\delta}{Nk} |\Omega_T^2|,$$

where we let

$$(7.39) \quad \epsilon_i \leq (\text{diam}(U'))^{-1} \frac{\delta}{2Nk} |\Omega_T^2|.$$

Having this choice for all $i \in \mathcal{I}_2$, we get

$$(7.40) \quad \sum_{i \in \mathcal{I}_2} \int_{Q_{i,1}} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \leq \frac{\delta}{k} |\Omega_T^2|.$$

Let $i \in \mathcal{I}_2$ and $(x, t) \in Q_{i,2}$; then $(\varphi_i)_x(x, t) \in \{-\lambda_{i,1}, \lambda_{i,2}\}$. Suppose $(\varphi_i)_x(x, t) = -\lambda_{i,1}$. By (7.11), we can choose a number $\bar{s} \in [s_{r_1}^-, s_{r_2}^-]$ such that

$$|(s_i - \lambda_{i,1}, \gamma_i) - (\bar{s}, \sigma(\bar{s}))| = \frac{\delta}{k}.$$

Then it follows from (a), (e), (7.7) and the previous equality that at the point (x, t) ,

$$\begin{aligned} & \left| \begin{pmatrix} (u_\eta)_x & (u_\eta)_t \\ (v_\eta)_x & (v_\eta)_t \end{pmatrix} - \begin{pmatrix} \bar{s} & (u_\eta)_t \\ u_\eta & bu_\eta + \mathcal{P}_{u_\eta} + F + \sigma(\bar{s}) \end{pmatrix} \right| \\ &= |((u_\eta)_x, (v_\eta)_t - bu_\eta - \mathcal{P}_{u_\eta} - F) - (\bar{s}, \sigma(\bar{s}))| \\ &\leq |(u_x, v_t - bu - \mathcal{P}_u - F) - (s_i, \gamma_i)| \\ &\quad + |(s_i - \lambda_{i,1}, \gamma_i) - (\bar{s}, \sigma(\bar{s}))| + |(\psi_i)_t| + |b||\varphi_i| \\ &\quad + |d(t)| \left| \int_{x_{i,1}}^x \varphi_i(y, t) \, dy \right| \\ &\leq \frac{\delta}{4k} + \frac{\delta}{k} + \epsilon_i(1 + \|b\|_\infty + \|d\|_{L^\infty(0,T)}) \leq \frac{3\delta}{2k}, \end{aligned}$$

where we let ϵ_i satisfy (7.23). The same can be shown in the case that $(\varphi_i)_x(x, t) = \lambda_{i,2}$; we omit the details. Thus we get

$$\int_{Q_{i,2}} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \leq \frac{3\delta}{2k} |Q_{i,2}|,$$

and so

$$(7.41) \quad \sum_{i \in \mathcal{I}_2} \int_{Q_{i,2}} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \leq \frac{3\delta}{2k} |\Omega_T^2|.$$

Gathering all of (7.3), (7.6), (7.10), (7.40) and (7.41), we obtain

$$\begin{aligned} & \int_{\Omega_T^2} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \\ &= \int_{\Omega_T^2 \setminus \bar{G}} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dxdt \\ &+ \int_{G \setminus (\cup_{i=1}^N \bar{Q}_i)} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dxdt \\ &+ \int_{\cup_{i \in \mathcal{I}_1} Q_i} \text{dist}(\nabla w(x, t), K((x, t); u)) \, dxdt \\ &+ \sum_{i \in \mathcal{I}_2} \int_{Q_{i,1}} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \\ &+ \sum_{i \in \mathcal{I}_2} \int_{Q_{i,2}} \text{dist}(\nabla w_\eta(x, t), K((x, t); u_\eta)) \, dxdt \\ &\leq \left(\frac{\delta}{k} + \frac{\delta}{k} + \frac{5\delta}{4k} + \frac{\delta}{k} + \frac{3\delta}{2k} \right) |\Omega_T^2| = \frac{23\delta}{4k} |\Omega_T^2| < \delta |\Omega_T^2|, \end{aligned}$$

where we let $k \geq 6$; hence, (7.38) is satisfied.

Substep 4-5. In conclusion, let us take an $l \in \mathbb{N}$ satisfying (7.37), choose a $k \in \mathbb{N}$ to fulfill $k \geq 6$ and (7.32) and numbers $\epsilon_i > 0$ ($i \in \mathcal{I}_2$) so that (7.15), (7.19), (7.23), (7.30) and (7.39) are satisfied. Then we have (7.13), (7.14), (7.16), (7.17), (7.18), (7.24) and (7.38); that is, the function $w_\eta = (u_\eta, v_\eta)$ indeed fulfills all the desired properties in (7.2).

The proof of Lemma 4.1 is now complete.

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